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RESTORATION OF CONTROLS UNDER CONDITIONS OF INCOMPLETE INFORMATION ON THE DYNAMICS OF THE SYSTEM†

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The inverse dynamical problem of the restoration of the controls or parameters of a dynamical system which are unknown in advance is considered, using results from the observation of the motion of the system under conditions when there is incomplete information on the phase states of the system. It is assumed that, at appropriate actual instants of time, the observer only obtains certain information sets containing the actual phase states of the system. It is well known that this problem is ill posed. Constructive dynamic regularizing algorithms for solving the problem are constructed which possess the property of physical feasibility and are capable of working under real-time conditions while processing incoming information during the motion of the system and producing a result in dynamics as the motion develops. © 1998 Elsevier Science Ltd. All rights reserved.

In this paper we use the ideas of positional control, under conditions where there is incomplete information, from the theory of differential games [1–6], (see also [7–13]). The previously proposed approach to inverse problems in dynamics [14–17] is the basis for solving the problem.

1. FORMULATION OF THE PROBLEM

Consider a controllable dynamical system, the behaviour of which in a specified time interval $T = [t_0, \vartheta]$ $(-\infty < t_0 < \vartheta < +\infty)$ is described by the system of ordinary differential equations

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t_0 \leqslant t \leqslant \vartheta; \quad x(t) \in R^n, \quad u(t) \in R^m \tag{1.1}$$

Here, $x(t)$ is the vector of the state of the system at the instant of time $t \in T$ and $u(t)$ is the vector of the control actions at this instant of time. The value of $u(t)$ is selected within the limits of the non-empty compact set $P(t) \subset R^m$, the compact sets $P(t)$ change continuously in the Hausdorff metric as the time $t \in T$ changes. The function f is continuous in $T \times R^n \times R^m$ and satisfies the usual conditions (the condition of sublinear growth and a local Lipschitz condition with respect to the variable x; see [2, 18, 19], for example) that ensure the existence of a unique solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ which is absolutely continuous in T for any initial position $(t_0, x_0) \in T \times R^n$, and any permissible control $u(\cdot) \in U$. The non-empty set of permissible controls U consists of all Lebesgue measurable functions $u(\cdot): T \ni t \to u(t) \in P(t)$. The solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ is sometimes called the motion of the dynamical system (1.1) generated by a control $u(\cdot)$ from the initial position (t_0, x_0).

Suppose that some bounded set of initial states $X_0 \subset R^n$ is specified and

$$X = \{x(\cdot) = x(\cdot\,; t_0, x_0, u(\cdot)) : x_0 \in X_0, u(\cdot) \in U\}$$

For each possible motion $x(\cdot) \in X$, we denote the set of all permissible controls which generate the given motion

$$U(x(\cdot)) = \{u(\cdot) \in U : x(\cdot) = x(\cdot\,; t_0, x_0, u(\cdot))\}$$

by $U(x(\cdot))$.

For any many-valued mapping $G(\cdot)$: $T \to \text{comp}(R^n)$, where $\text{comp}(R^n)$ is the set of all non-empty compact sets from R^n, suppose $X[G(\cdot)]$ denotes the set of all possible motions of system (1.1) lying in the mapping (in the phase constraints) $G(\cdot)$

$$X[G(\cdot)] = \{x(\cdot) \in X : x(t) \in G(t), t \in T\}$$

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where $U[G(\cdot)]$ is the set of all controls generating the motions from $X[G(\cdot)]$, and

$$U[G(\cdot)] = \cup\{U(x(\cdot)) : x(\cdot) \in X[G(\cdot)]\}$$

We will now give a meaningful description of the problem. We will assume that a certain motion $x_*(\cdot) \in X$ is controlled in a time interval T. During this process, the observer obtains certain information which only enables him to estimate certain sets $G(t)$ in the phase space of the system, which contain the actual states $x_*(t)$ of the motion $x_*(\cdot)$. However, this information is insufficient either for an exact calculation of the value of $x_*(t)$ or for its satisfactory statistical description within the limits of this information set $G(t)$. The question as to how the sets $G(t)$ are formed using some method of observing the system and the motion is left to one side for the moment. We simply assume that the information arriving at the observer up to an instant of time $t \in T$ enables him to determine the set $G(t)$ while, at the same time, it is possible for him to refine these data at the instant of time t in order to find a smaller subset of the set $G(t)$. Some of the methods of observation are described in [1, 4], for example.

The set $G(t)$ is naturally treated as an "ideal" result of observation. By virtue of the inevitable noise in the observation and measurement channel, this result is unattainable in practice. The actual result of observation is to be considered as a certain "perturbed" set $Z(t) \in \text{comp}(R^n)$, which differs from the set $G(t)$ in the magnitude of the error with respect to a certain observation error criterion $\nu : \text{comp}(R^n) \times \text{comp}(R^n) \to R_+ = [0, \infty)$

$$\nu(Z(t), G(t)) \leqslant h, \quad t \in T$$

Let us agree to call the function $G(\cdot) : T \ni t \to G(t) \in \text{comp}(R^n)$ the ideal result of observation while we call the function $Z(\cdot) : T \ni t \to G(t) \in \text{comp}(R^n)$ the h-perturbation of the ideal result of observation $G(\cdot)$.

Suppose a certain criterion of the closeness of the controls $\rho : U \times U \to R_+$ is given. The problem being considered can be formulated in advance as follows: it is required to construct an algorithm which, in the dynamics with respect to an h-perturbation $Z(\cdot)$ of the ideal result of observation $G(\cdot)$, constructs a control $u_h(\cdot) \in U$ which, for a sufficiently small $h > 0$, is sufficiently close, with respect to the criterion ρ, to one of the controls of the set $U(x_*(\cdot))$ or, in the extreme case, of the set $U[G(\cdot)]$. It is assumed that information concerning the dynamics of system (1.1) and the sets $P(t)$, $t \in T$ is known in advance by the observer who is seeking to solve the restoration problem which has been formulated.

The restoration problem described is a version of the formulation of a well-known inverse problem of dynamics for control systems [20–23]. The special feature of the formulation considered here is the following: first, the problem has to be solved under conditions when the information on the actual phase positions of the dynamical system is substantially incomplete, second, the required algorithm must work in real time and possess the property of physical feasibility and, third, the algorithm must be stable with respect to small perturbations of the ideal result of observation.

We will now discuss the special features of the formulation of the problem and a method for solving it. It is clear that the incompleteness of the information considerably reduces the ability of the observer to restore the unknown control. Therefore, in order just to obtain some solution of the problem, we assume that some law of evolution of the information sets $G(t)$, $t \in T$ is also known. In solving the problem, we shall seek a method of solving it which will realize the restoration of the required control in the dynamics synchronously with the development of the process in time or, as is sometimes also said, at the rate of real time. In restoring the required quantity, the observer can only take account of that information which has arrived at the corresponding instant of time. The restoration process must be a "one-shot" process as it would be impossible to repeat it without going backwards in time. In order that the dynamic solution should be of practical value, it is necessary that the corresponding solution operations should be constructed in the class of operations with the property of physical feasibility which is sometimes referred to as a hereditary property or a causality property: the results of operations (outputs) coincide in time as long as the arguments (outputs) coincide in time [2, 18, 19].

The corresponding justifications and various examples of meaningful problems, in which it is important to obtain a dynamic solution of an inverse problem, are presented in [14, 17, 24], for example. When there is noise in the observation channel, the problem may turn out to be ill-posed and the corresponding solution operation must therefore also possess regularizing properties [25–27]. In estimating the possibility of a practical solution of the problem using a computer, we shall try to find its solution using a scheme which is discrete in time.

We will now refine the formulation of the problem. On considering, *a posteriori*, the inflow of information during the observation $G(\cdot)$, the observer is forced to conclude that, in principle, any control from the set $U[G(\cdot)]$ can be compatible with this information since each control $u(\cdot)$ from this set, in the case of a certain initial state $x_0 \in G(t_0)$, generates a motion of the system $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ which satisfies the inclusion $x(t) \in G(t), t \in T$. It is necessary that the actual control $u_*(\cdot)$ which generates the motion $x_*(\cdot)$ belongs to the set $U(x_*(\cdot))$ and, all the more, to the set $U[G(\cdot)]$ but, generally speaking, it has to be guessed as to which of the controls from the set $U[G(\cdot)]$ and, all the more, from the set $U(x_*(\cdot))$ is $u_*(\cdot)$ as the observer cannot, even if the information $G(\cdot)$ is exact. Generally speaking, the motion $x_*(\cdot)$ will not be known *a posteriori* as all the information concerning the motion is used up (when there are no errors) towards the end of the time interval T just by the observation $G(\cdot)$. Hence, there is an equal chance that each of the motions $x(\cdot) \in X[G(\cdot)]$ refers to the actual motion.

Hence, every control $u(\cdot) \in U[G(\cdot)]$ with the same *a posteriori* chances may refer to the actual motion. At the same time, it is automatic that every control outside the bounds of the set $U[G(\cdot)]$ cannot be the actual control. In other words, $U[G(\cdot)]$ has the meaning of the set of those and only those controls which are not precluded *a posteriori* as the actual control. It is also natural to attempt to find the controls from this set in the first place. When there is additional information on the behaviour of the sets $G(t)$, $t \in T$, one can also attempt to find the controls from the set $U(x_*(\cdot))$.

Suppose a certain criterion for the selection of controls, which is defined by the functional $\omega: U \to R$, is specified in U and $U_*[G(\cdot)]$ is a subset of the set $U[G(\cdot)]$, that consists of the elements which satisfy this criterion

$$U_*[G(\cdot)] = \{u(\cdot) \in U[G(\cdot)] : \omega(u(\cdot)) \leq \omega_*\}$$

where ω_* is a certain specified fixed number. Suppose $\Xi[h, G(\cdot)]$ is the set of all many-valued mappings $T \to \text{comp}(R^n)$ which can aspire to the role of h-perturbation of the ideal result of the observation $G(\cdot)$

$$\Xi[h, G(\cdot)] = \{Z(\cdot) \in (T \to \text{comp}(R^n)) : \nu(Z(t), G(t)) \leq h, \quad t \in T\}$$

When there is noise in the observation channel, the observer can, in fact, obtain any of the elements of the set $\Xi[h, G(\cdot)]$ as the h-perturbation of the ideal result of observation. It is therefore natural that the method or restoration algorithm should be calculated on receipt of any element from $\Xi[h, G(\cdot)]$.

The simplest examples show that the problem being considered can turn out to be unstable with respect to small perturbations of the sets $G(t)$, $t \in T$. The restoration algorithm must therefore be stable, that is, the result of the operation of the algorithm must be as close as desired to some element of the set $U_*[G(\cdot)]$ or to this same set according to the criterion of the closeness of the controls ρ for a sufficiently small h whatever the h-perturbation $Z(\cdot) \in \Xi[h, G(\cdot)]$ is here.

The property of the physical feasibility of the restoration algorithm (or, what is the same thing, of the corresponding operator $D : R_+ \times (T \to \text{comp}(R^n)) \to U$) signifies the following: $u_1(t) = u_2(t)$, $t_0 \leq t \leq \tau$ only if $u_1(\cdot) = D(h, Z_1(\cdot))$, $u_2(\cdot) = D(h, Z_2(\cdot))$, $h > 0$, $Z_1(t) = Z_2(t)$, $t_0 \leq t \leq \tau$, $t_0 \leq t \leq \vartheta$. Every algorithm (operator) D, which calculates its values using a positional method, that is, $u(t)$ for $t \in T$ is determined using t, $Z(t)$ and, possibly, certain auxiliary internal variables, automatically possesses the above-mentioned property.

The restoration problem can now be formulated as follows: it is required to construct an operator $D : R_+ \times (T \to \text{comp}(R^n)) \to U$, possessing the property of physical feasibility and such that, for any fixed $G(\cdot) \in M \subseteq (T \to \text{comp}(R^n))$, the condition

$$\sup\{\rho(D(h, Z(\cdot)), U_*[G(\cdot)]) : Z(\cdot) \in \Xi[h, G(\cdot)]\} \to 0, \quad h \to 0$$

is satisfied, where M is a certain set of possible ideal results of observations which are subject to appropriate processing and

$$\rho(D(h, Z(\cdot)), U_*[G(\cdot)]) = \inf\{\rho(D(h, Z(\cdot)), u(\cdot)) : u(\cdot) \in U_*[G(\cdot)]\}$$

An element $u_h(\cdot) = D(h, Z(\cdot))$ can be taken as being close to a certain element of an *a priori* unknown set of controls $U_*[G(\cdot)]$. In order to construct the restoration algorithm D we shall seek a suitable positional strategy V of control by a certain auxiliary system-model. The realization of the strategy V will also be taken as the value of the algorithm D. The system-model must be close in a certain sense to the initial system and the strategy V must be such that the motions of the system-model which it generates track the evolution of the information sets in a certain sense. In the following section, we refine all of the concepts and informally describe a method for solving the restoration problem.

2. SOLUTION OF THE PROBLEM

We will introduce a system-model into the treatment which, for simplicity, is a copy of the initial system

$$\dot{y}(t) = f(t, y(t), v(t)), \quad t_0 \leqslant t \leqslant \vartheta \tag{2.1}$$

Here, the control actions are constrained by the condition $v(t) \in P(t)$, $t \in T$.

The positional strategy V of the control by the model is identified with the mapping $T \times T \times R^n \times \operatorname{comp}(R^n) \to \Sigma$, where Σ is the set of all contractions of the functions from U in all possible half intervals $[t, s) \subset T$, $\Sigma = \cup\{U[t, s) : [t, s) \subset T\}$ and $U[t, s)$ is the set of all Lebesgue measurable functions $u(\cdot) : [t, s) \ni t \to u(t) \in P(t)$. For arbitrary $t \in T$, $s \in T$, $y \in R^n$, $Y \in \operatorname{comp}(R^n)$, we assume that $V(t, s, y, Y)$ is an arbitrary element of U if $t \geqslant s$ or $y \in Y$; if $t < s$ and $y \notin Y$, then $V(t, s, y, Y)$ is any of the elements $v(\cdot) \in U[t, s)$ which satisfies the condition

$$\left\langle y - z, \int_t^s f(\tau, y, v(\tau))d\tau \right\rangle \leqslant \inf\left\{\left\langle y - z, \int_t^s f(\tau, y, w(\tau))d\tau \right\rangle : w(\cdot) \in U[t, s)\right\} + (s-t)^2$$

where $\langle \cdot, \cdot \rangle$ is a scalar product R^n and z is any of the vectors of the set Y which is closest in the Euclidean metric to the vector y. The above-mentioned strategy is an analogue of the well-known extremal strategy from the theory of differential games [1–5, 19].

We will say that the mapping $G(\cdot) : T \to \operatorname{comp}(R^n)$ is stable by virtue of system (2.1) if, for any $t_1 \in T$, $t_2 \in T$, $t_1 < t_2$, $y_1 \in G(t_1)$, a control $w(\cdot) \in U[t_1, t_2)$ exists such that the solution $y(\cdot) = y(\cdot\,; t_1, y_1, w(\cdot))$ of the Cauchy problem $\dot{y}(t) = f(t, y(t), w(t))$, $y(t_1) = y_1$, $t_1 \leqslant t, \leqslant t_2$ satisfies the condition $y(t_2) \in G(t_2)$. The set S of all mappings which are stable by virtue of system (2.1) is a non-empty set. Suppose that S_0 is a subset of the set S for each element $G(\cdot)$ of which a number $C > 0$ is found such that, for any $h > 0$, $t \in T$, $y \in R^n$, $Z(\cdot) \in \Xi[h, G(\cdot)]$, $y_1 \in Q(y, G(t))$, $y_2 \in Q(y, Z(t))$ the inequality $\|y_1 - y_2\| \leqslant Ch$ is satisfied, where $\|\cdot\|$ is the Euclidean norm in R^n and $Q(y, \Omega)$ is the set of elements of the compact set $\Omega \subset R^n$ which is closest in the Euclidean metric to the element $y \in R^n$. S_0 is a non-empty set.

We will now describe the restoration algorithm and fix the arbitrary $h > 0$, $G(\cdot) \in (T \to \operatorname{comp}(R^n))$, $Z(\cdot)\, \Xi[h, G(\cdot)]$, the subdivision Δ of the interval T by the points t_i, $t_0 < t_1 < \ldots < t_m = \vartheta$, the dependence $m = m(h)$ and the number $C_* > 0$ such that

$$\operatorname{diam} \Delta = \max\{t_{i+1} - t_i : i = 0, \ldots, m-1\} \leqslant C_* h$$

We now consider a control $u_h(\cdot) \in U$ which is formed using the rule

$$u_h(t) = v_i(t), \quad t_i \leqslant t < t_{i+1}, \quad v_i(\cdot) = V(t_i, t_{i+1}, y(t_i), Z(t_i))$$

where $y(t_i)$ is the state of the model at the instants of time t_i, $i = 0, \ldots, m-1$. The transition from one state of the model to another is made in accordance with the differential equation for the motion of the model

$$y(t_i) = y(t_{i-1}) + \int_{t_{i-1}}^{t_i} f(\tau, y(\tau), v_i(\tau))d\tau$$

where $y(t_0) = y_0$ is an arbitrary fixed initial state of the model from the set $Z(t_0)$.

We now define the operator (algorithm) D according to the rule

$$D(h, Z(\cdot)) = u_h(\cdot) \tag{2.2}$$

We will describe the operation of this algorithm in time. Up to the instant of time t_0, depending on the level of the error h, a subdivision Δ of the interval T is chosen and fixed which satisfies the condition $\Delta \leqslant C_* h$. Each point t_i, used in the subdivision Δ, will be the start of the following step in calculating the new state of the model and a new form of the strategy V. At the instant of time t_0, an information set $Z(t_0)$ is received by the observer from which he selects and fixes some initial state y_0 for the model. The form $v_0(\cdot) = Vt_0)$, $t_1, y_0, Z(t_0))$ of the strategy V in the interval $t_0 \leqslant t < t_1$ an the state $y(t_1)$ of the model at the instant of time t_1 are then calculated. At the instant t_1, an information set $Z(t_1)$ is received by the observer which is used, together with the state of the model $y(t_1)$, to find the form $v_1(\cdot) = V(t_1, t_2, y(t_1), Z(t_1))$ of the strategy V in the interval $t_1 \leqslant t < t_2$ and the state $y(t_2)$ of the model at the instant t_2. The corresponding new forms of the strategy and the states of the model are constructed in the following intervals on receipt of new information sets by the observer, by analogy with those which have been constructed in the preceding step. Towards the final instant of time ϑ, the realization of the algorithm $u_h(\cdot)$ will be formed in the dynamics which is also taken as an approximation to the required controls. From the description of the operation of the algorithm in time it is clear that its realization in real time is also possible. The variable describing the state of the model can be considered as an internal variable of the algorithm, and the values of this variable can be physically worked out completely autonomously using a computer.

We will now point out certain conditions under which the algorithm automatically provides a solution of the restoration problem.

Condition 1: (a) ν is the Hausdorff metric in comp(R^n); (b) (U, ρ) is a compact metric space; (c) $\omega_* > \sup\{\omega(u(\cdot)) : u(\cdot) \in U\}$; (d) $M = S_0$; *e*) if $\rho(u_k(\cdot), u(\cdot)) \to 0$, then $x(\cdot; t_0, x_0, u_k(\cdot)) \to x(\cdot\,; t_0, x_0, u(\cdot))$ $C(T; R^n)$.

We will comment on this condition. The Hausdorff metric is often used to estimate the distances between compact sets [1–6, 18, 19, 28, 29]. The space (U, ρ) is compact, for example, when U is a weakly compact set in $L^p(T; R^m)$, $1 < p < \infty$ and ρ is a so-called "weak" metric in U [18]. In this case, condition 1*c* in fact precludes an additional selection of controls from the problem as, in this case, $U_*[G(\cdot)] = U[G(\cdot)]$. However, there will actually be an additional selection of controls in one form of the problem under consideration which will be discussed below. The condition $M = S_0$ arises from the desired to use stable external constructions from the theory of positional control to solve the inverse problem in dynamics [1–5, 19]. The last property from the condition is automatically satisfied, for example, in the case of linear systems or non-linear systems with a right-hand side of the form $f = f_1(t,x)u + f_2(t,x)$.

Theorem 1. When condition 1 is satisfied, the dynamic algorithm (2.2) gives the solution to the restoration problem.

Proof. The property of the physical feasibility of the dynamic algorithm (2.2) follows from the positional character of the strategy V. To prove the theorem, it is now sufficient to show that, whatever the element $G(\cdot) \in M$ and whatever the sequences $\{h_k\}$ $(h_k > 0, h_k \to 0)$, $\{Z_k(\cdot)\}$ $(Z_k(\cdot) \in \Xi[h_k, G(\cdot)])$, the convergence $\rho(u_k(\cdot), U_*[G(\cdot)]) \to 0$ when $k \to \infty$ will hold for the controls $u_k(\cdot) = D(h_k, Z_k(\cdot))$. On taking account of the definition of the strategy V and the rule for the formation of the controls $u_k(\cdot)$, the following estimate can be obtained for the error $\varepsilon_k[t] = \min\{\|y(t; t_0, y_0, u_k(\cdot)) - g\|^2 : g \in G(t)\}$

$$\max\{\varepsilon_k[t] : t \in T\} \leq C_0 h_k$$

where C_0 is a certain positive number which is independent of the number k and is defined solely *a priori* by the known data concerning the problem. It follows from this that $\varepsilon_k[t] \to 0$ as $k \to \infty$ for each $t \in T$. On taking account of compactness of the space (U, ρ), it can be assumed without loss of generality that the convergence $\rho(u_k(\cdot), u_0(\cdot)) \to 0$ holds in the case of a certain control $u_0(\cdot) \in U$. Then, $y(\cdot\,; t_0, y_0, u_k(\cdot)) \to y(\cdot\,; t_0, y_0, u_0(\cdot))$ in $C(T; R^n)$ and, by virtue of the completeness of the sets $G(t)$, $t \in T$, we have $y(t; t_0, y_0, u_0(\cdot)) \in G(t)$ for each $t \in T$. It follows from this that $u_0(\cdot) \in U_*[G(\cdot)] = U[G(\cdot)]$ and, therefore, $\rho(u_k(\cdot), U_*[G(\cdot)]) \to 0$. The theorem is proved.

3. A PRECISE STATEMENT OF THE PROBLEM

We will now consider the following version of the problem. Suppose the control system is linear

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t), \quad t_0 \leq t \leq \vartheta \tag{3.1}$$

where $A(t)$, $B(t)$, $F(t)$ are matrices with the dimensions $n \times n$, $n \times m$, $n \times 1$, respectively with elements which are continuous in T and the sets $P(t)$ are convex compact sets in R^m which depend continuously on $t \in T$.

Note that the dynamics of the convex compact sections $X(t) = \{x(t; t_0, x_0, u(\cdot)) : x_0 \in X_0\}$, $t_0 \leq t \leq \vartheta$ and of the pencil of motions $X(\cdot\,; t_0, x_0, u(\cdot)) = \{x(\cdot\,; t_0, x_0, u(\cdot)) : x_0 \in X_0\}$ of system (3.1), which has emerged from the convex compact set X_0 under the control $u(\cdot)$ at the instant t_0, are described using the support functions with the equality

$$\sigma(l, K(\vartheta, t)X(t)) = \sigma(l; K(\vartheta, t_0)X(t_0)) + \int_{t_0}^{t} \langle K(\vartheta, \tau)[B(\tau)u(\tau) + F(\tau)], l\rangle d\tau$$

where $K(\cdot, \cdot)$ is the Cauchy matrix of linear system (3.1)

$$\partial K(t,\tau)/\partial t = A(t)K(t,\tau), \quad K(\tau,\tau) = E$$

E is a unit matrix and $\sigma(l; W) = \sup\{\langle l, w\rangle : w \in W\}$ is the support function of the set $W \subset R^n$, $l \in L = \{l \in R^n : || \leqslant 1\}$.

We will now specify the nature of the change in the information sets $G(t)$ with time in the given precise definition. We assume that, whatever the instants of time $t_1, t_2 \in T$, $t_1 \leqslant t_2$, a certain permissible form of the control action $u(\cdot) \in U[t_1, t_2)$ is found, by means of which each point of the set $G(t_2)$ can be obtained by passing, by virtue of system (3.1), from a certain point of the set $G(t_1)$. In other words

$$G(t_2) \subseteq X(t_2; t_1, G(t_1), u(\cdot)) = \{x(t_2; t_1, x_1, u(\cdot)) : x_1 \in G(t_1)\}$$

When the information sets evolve in such a way, it can be assumed that the set $G(t)$ is a section at the instant t of a certain pencil of the system motions which emerges at the instant t_0 from a certain set $G^*(t) \subseteq G(t_0)$ due to the action of a certain permissible control. It is clear that $G^*(t_0) = G(t_0)$ $G^*(t_2) \subseteq G^*(t_1)$ when $t_2 \geqslant t_1$. Suppose the mapping $\Gamma : T \times \mathrm{sub}(R^n) \to \mathrm{sub}(R^n)$, where $\mathrm{sub}(R^n)$ is the set of all non-empty subsets of the set R^n defines the law according to which the refinement which time of the set of initial states is attained. The law Γ must satisfy the natural conditions $\Gamma(t_0, N) = N$ and $\Gamma(t_2, N) \subseteq \Gamma(t_1, N)$ when $t_2 \geqslant t_1$. The dynamics of the information sets can be represented in the form

$$K(\vartheta, t)G(t) = \Gamma(t, K(\vartheta, t_0)G(t_0)) + \int_{t_0}^{t} K(\vartheta, \tau)[B(\tau)u(\tau) + F(\tau)]d\tau$$

Henceforth, we will only consider compact and convex information sets. The set of all non-empty convex compact sets from R^n is denoted by $\mathrm{cconv}(R^n)$. A convex compact set is fully characterized by its support function. The set of all mappings $G(\cdot) : T \to \mathrm{cconv}(R^n)$, the evolution of which with time satisfies the conditions which have been described above, while the law Γ for the refinement of the initial data satisfies a Lipschitz condition with respect to the second variable in relation to the criterion ν, is denoted by S_1. This is a non-empty set.

For each ideal result of observation $G(\cdot) \in S_1$, it is natural to consider a set of permissible controls $U^0[G(\cdot)] \subseteq U[G(\cdot)]$, each of which, in a pair with some permissible law of refinement of the initial data, generates a given observation $G(\cdot)$. It is clear that $U^0[G(\cdot)] \subseteq U[G(\cdot)]$. However, it is somewhat better to find some control from $U^0[G(\cdot)]$ than to find some control from $U[G(\cdot)]$ as each control from $U^0[G(\cdot)]$ generates a certain pencil of motions which emerge at the instant t_0 from a certain convex subcompact set of the compact set $G(t_0)$ and is contained within the bounds of $G(t)$, $t_0 \leqslant t \leqslant \vartheta$ and, consequently, possesses a certain universality with respect to the initial data. We shall also attempt to find the controls from $U^0[G(\cdot)]$.

We introduce a system-model into the treatment, which is a copy of the initial system (3.1). The vector of the control actions in the model satisfies the same constraints as in the initial system. Using the system-model, we construct the appropriate pencils of motions and identify the positional strategy V of the control by the model with the mapping $T \times T \times \mathrm{cconv}(R^n) \times \mathrm{cconv}(R^n) \to \Sigma$. In the case of arbitrary $t \in T$, $s \in T$, $Y_1 \in \mathrm{cconv}(R^n)$, $Y_2 \in \mathrm{cconv}(R^n)$, we assume that $V(t, s, Y_1, Y_2)$ is an arbitrary element of U, if $t \geqslant s$ while, if $t < s$, then $V(t, s, Y_1, Y_2)$ is the single element $v(\cdot) \in U[t, s)$, which makes the quadratic functional

$$H(v) = 2\langle\langle \sigma(l; K(\vartheta, t_1)Y_1) - \sigma(l; K(\vartheta, t_1)Y_2), \int_t^s \langle K(\vartheta, \tau)B(\tau)v(\tau), l\rangle d\tau\rangle\rangle + \alpha(h)\int_t^s \|v(\tau)\|^2\, d\tau$$

a minimum in $U[t, s)$. In this functional, $\langle\langle \cdot, \cdot \rangle\rangle$ is a scalar product in $L^2(L; R)$ ($\|\cdot\|_*$ is the norm in this space) and $\alpha(\cdot)$ is some positive function in R_+ which satisfies the condition $\alpha(h) \to 0$, $h/\alpha(h) \to 0$ as $h \to 0$.

We will now describe the restoration algorithm, assuming that the law for the refinement of the initial data is known to the observer. We fix arbitrary $G(\cdot) \in S_1$, $h > 0$, $Z(\cdot) \in \Xi[h, G(\cdot)]$ and the subdivision Δ of the interval T with diam $\Delta \leqslant C_* h$ and consider a control $u_h \in U$ which is formed according to the rule

$$u_h(t) = v_i(t), \quad t_i \leqslant t < t_{i+1}, \quad v_i(\cdot) = V(t_i, t_{i+1}, Z(t_i), Y(t_i))$$

where $i = 0, \ldots, m-1$ $Y(t_i)$ is section of the pencil of motions of the model at the instant t_i and

$$K(\vartheta,t_i)Y(t_i)=\Gamma(t_i,K(\vartheta,t_0)Z(t_0))+\int_{t_0}^{t_i}K(\vartheta,\tau)[B(\tau)u_h(\tau)+F(\tau)]d\tau$$

We now define the operator (algorithm) D according to the rule

$$D_1(h,Z(\cdot))=u_h(\cdot) \tag{3.2}$$

The operation of this algorithm with time is analogous to the operation of the similar algorithm from Section 2.

We will now indicate certain conditions under which algorithm (3.2) automatically provides the solution of the present problem of the restoration of controls from the set $U^0[G(\cdot)]$ for $G(\cdot)\in S_1$.

Condition 2: (a) ν is the metric in cconv(R^n) which is determined by the equality $\nu(N_1, N_2) = \|\sigma(\cdot;N_1)-\sigma(\cdot\,N_2)\|_*$; (b) ρ is the metric of the space $L^p(T;R^m)$, $1\leqslant p<\infty$; (c) ω is the norm of the space $L^2(T;R^m)$ and $\omega_*=\inf\{\omega(u(\cdot)):u(\cdot)\in U^0[G(\cdot)]\}$; (d) $M=S_1$; (e) the law for refining the initial data for each process which is observed is known to the observer.

Theorem 2. When condition 2 is satisfied, the dynamic algorithm (3.2) provides a solution of the restoration problem: for any fixed $G(\cdot)\in M$, the condition

$$\sup\{\rho(D_1(h,Z(\cdot)),U_*[G(\cdot)]):Z(\cdot)\in\Xi[h,G(\cdot)]\}\to 0,\quad h\to 0$$

is satisfied, where $U_*[G(\cdot)]=\{u(\cdot)\in U^0[G(\cdot)]:\omega(u(\cdot))\leqslant\omega_*\}$.

Proof. The property of the physical feasibility of dynamic algorithm (3.2) follows from the positional nature of the strategy V. The assertion of the theorem will now follow from the fact that, for each fixed $G(\cdot)\in M$ in the case of any sequences $\{h_k\}\subset R_+(h_k\to 0)$, $\{Z_k(\cdot)\}$ $(Z_k(\cdot)\in\Xi[h_k, G(\cdot)])$, the convergence $\rho(u_k(\cdot),u_0(\cdot))\to 0$ when $k\to\infty$, where $u_0(\cdot)$ is the single element of which the set $U_*[G(\cdot)]$ consists, will hold for the controls $u_k(\cdot)=D_1(h_k,Z_k(\cdot))$. Note that, generally speaking, the set $U^0[G(\cdot)]$ can consist of several elements and is a convex bounded and closed set in $L^2(T;R^m)$. Therefore, after the elements in the set $U_*[G(\cdot)]$ have been selected using the criterion ω, just a single element of the set $U^0[G(\cdot)]$ must remain with the minimal $L^2(T;R^m)$-norm.

On taking the rule for the formation of the control into account, it is possible to obtain the following estimate for the functional Λ_k

$$\max\{\Lambda_k[t]:t\in T\}\leqslant C_0h_k$$

where C_0 is a certain positive number which is independent of the number k and is solely determined *a priori* by the known data appertaining to the problem

$$\Lambda_k[t]=\|\sigma(\cdot;K(\vartheta,t)Y_k(t))-\sigma(\cdot;K(\vartheta,t)G(t))\|_*^2+\alpha(h)\int_{t_0}^{t}[\|u_k(\tau)\|^2-\|u_0(\tau)\|^2]d\tau$$

The two estimates

$$\max\{\|\sigma(\cdot;K(\vartheta,t)Y_k(t))-\sigma(\cdot;K(\vartheta,t)G(t))\|_*^2:t\in T\}\leqslant C_0h_k+2\alpha(h)b(\vartheta-t_0)$$

$$(b=\sup\{\|w\|^2:w\in P(t),\ \ t\in T\}<\infty)$$

$$\omega(u_k(\cdot))\leqslant\omega(u_0(\cdot))+C_0h_k/\alpha(h_k)$$

follow from the above-mentioned estimate of the functional Λ_k.

It follows from the last two estimates that $\omega(u_k(\cdot))\to\omega(u_0(\cdot))$ and $u_k(\cdot)\to u_0(\cdot)$ weakly in $L^2(T;R^m)$. Hence, $u_k(\cdot)\to u_0(\cdot)$ strongly in $L^2(T;R^m)$. By virtue of the boundedness of the set U in $L^\infty(T;R^m)$, the convergence $u_k(\cdot)\to u_0(\cdot)$ also holds strongly in $L^p(T;R^m)$, $1\leqslant p<\infty$. The theorem is proved.

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